International Journal of Pure & Applied Mathematical Research, Voleme1, issue1, Jan-Jun -2017, pp 22-35 , ISSN: 2456-7493

PROLONGATION OF HYPERSURFACES WITH QUARTER SYMMETRIC METRIC CONNECTION TO TANGENT BUNDLE

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ABSTRACT

The present paper is to considering the Lifting theory, we study lift of hypersurfaces with quarter symmetric metric in connection to the tangent bundles and to obtain certain results on totally geodesic. Also we obtain structure equations with respect to quarter symmetric metric connection.

AMS Subject Classification: 53C05, 53C15, 53C22, 53B25, 53C40.

Keywords and phrases: Lift, hypersurfaces, tangent bundles, quarter symmetric metric connection.

1. Introduction

The differential geometry of tangent bundles of Riemannian manifolds has also been studied by Sasaki [10] and then Dombrocosk [2] in 1958 and 1962 respectively. In 1966, Yano and Kobayashi [12] studied the prologations of tensor fields and connection to tangent bundles. In [13, 14] Yano and Ishihara defined and studied prolongations called complete, vertical and horizontal lifts of tensor fields and connections. Mariko Tani [11], in 1969, had introduced the idea of prolongations of hypersurfaces to tangent bundles with respect to metric tensor which is complete lift of the metric tensor of original manifold.

In 1975, S. Golab [4] introduced the idea of quarter symmetric linear connection if its torsion tensor T is of the form

$$T(X,Y) = w(Y)FX - w(X)FY, \tag{1}$$

where w is a 1-form and F is a tensor field of the type $(1,1)$. In [7], R. S. Mishra and S. N. Pandey considerd a quarter symmetric symmetric F-connection and studied some of its properties. In [1] [8] and [9] some kind of quarter symmetric metric connection were studied. In 2012. C. Gozutok and A. Esin [3] has conducted study about the tangent bundles of hypersurfaces with semi-symmetric metric connection. Recently the lift of hypersurfaces with the help of quarter symmetric and semi metric connection to tangent bundle were resently discussed by N. I. Khan [6].

In the present paper, we study the prolongations of hypersurfaces to tangent bundles with quarter symmetric-metric connection. We consider the lifts of hypersurfaces with quarter symmetric metric connection to tangent bundles.

The paper is organized as follows: In Section 2, we give the neccessary details and results which requried in the next section. In Section 3, we show that the compelete lift of quarter symmetric metric connection on hypersurfaces is quarter symmetric metric connection in tangent bundle of the hypersurfaces and we also find some certain results concerning to the tangent bundles. In last Section, we obtain the structure equations with respect to quarter symmetric metric connection of tangent bundle.

2. Priliminaries

For the manifold M we denote by $T(M)$ it's tangent bundle with the projection $\pi_M : T(M) \to M$ and by $T_p(M)$ it's tangent space at the point p of M. $\Im_s^r(M)$ is the set of all tensor filed of type (r,s) in M. For example $\Im_0^0(M)$, $\Im_0^1(M)$, $\Im_1^0(M)$, $\Im_1^1(M)$ etc. are the set of functions, vector fileds, 1-forms, tensor fileds of the type $(1,1)$ etc. on M respectively.

Let X, w, F, g, T and R be a vector field a 1-form, tensor field of type $(1,1)$, of type $(1,2)$ and of type $(1,3)$ respectively. We denote respectively by X^V, w^V, F^V, g^V, T^V and R^V their vertical lifts by X^C, w^C, F^C, g^C, T^C and R^C their complete lifts.

Let ∇ be a linear connection in an n-dimensional diffrentiable manifold M. The torsion tensor T of ∇ and the curvature tensor R of ∇ are given by respectively

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \tag{2}$$

The connection ∇ is a symmetric if its torsion tensor T vanishes, or else it is non-symmetric. The connection ∇ is metric if there is a Riemannian metric g in M such that $\nabla g = 0$, otherwise it is non-metric. It is well know that a linear connetion is symmetric and metric if it is the Levi-Civita connection.

Let $\bar{\nabla}$ be a metric connection in (M,g), which is non-symmetric. In [4], if torsion tensor $\bar{T}$ of $\bar{\nabla}$ defined by (2.1), satisfies

$$T(X,Y) = w(Y)FX - w(X)FY, \tag{3}$$

for any $w \in \Im_1^0(M)$, then the connection $\bar{\nabla}$ is called quarter-symmetric metric connection in (M,g). A quarter-symmetric metric connection $\bar{\nabla}$ in (M,g) is given by

$$\overline{\nabla}_X Y = \nabla_X Y + w(Y)FX - g(FX, Y)P \tag{4}$$

for arbitrary vector fields X and Y in (M, g), where ∇ is a Riemannian connection in (M, g) and P is a vector field defined by $g(P, X) = w(X)$ for any vector field X in (M, g).

Now, according to [13], using the complete lift and vertical lift operations, we have the following equalities:

$$[X^C, Y^C] = [X, Y]^C, \tag{5}$$

$$w^V(X^C) = (w(X))^V,$$

$$w^C(X^C) = (w(X))^C$$

$$F^C(X^C) = F(X),$$

$$g^C(X^V, Y^C) = g^C(X^C, Y^V) = (g(X, Y))^V,$$

$$g^C(X^V, Y^C) = (g(X, Y))^C,$$

$$\nabla^C_{X^C} Y^C = (\nabla_X Y)^C,$$

$$\nabla^C_{X^C} Y^V = (\nabla_X Y)^V,$$

$$T^C(X^C, Y^C) = (T(X, Y))^C,$$

$$R^C(X^C, Y^C)Z^C = (R(X, Y)Z)^C$$

for any $X, Y, Z \in \Im^1_0(M)$, $w \in \Im^0_1(M)$, $F \in \Im^1_1(M)$, $g \in \Im^0_2(M)$, $T \in \Im^0_0(M)$ and $R \in \Im^1_3(M)$.

Let S be an $(n-1)$ -dimensional manifold embedded differentially as a submanifold in (M,g) and denote by $i:S\rightarrow M$ its embeddding. The differential mapping di is a mapping from TS into TM, which is called the tangent map of i, where TS and TM are tangent bundles of S and M, respectively. The tangent map of B is denoted by $B:T(TS)\rightarrow T(TM)$, where di is denoted by B.

The hypersurface S is also a Riemannian manifold with induced metric $\bar{g}$ defined by $\bar{g}=g(BX,BY)$ for arbitrary $X,Y\in\Im_0^1(M)$. Thus $\bar{\bar{\nabla}}$ is a Riemannian connection with induced conection on $(S,\bar{g})$ from ∇ defined by

$$\nabla_{BX}BY=B(\bar{\bar{\nabla}}_X Y)+h(X,Y)N \tag{6}$$

for any $X,Y\in\Im_0^1(M)$, where N is the unit normal vector field on (S,g) and h is the second fundamental tensor field of (S,g). Also the following relation holds on S

$$h(X,Y)=\bar{g}(HX,Y),$$

for any $X,Y\in\Im_0^1(M)$, where $H\in\Im_1^1(M)$.

If h vanishes, then S is called *totally geodesic* with respect to $\bar{\bar{\nabla}}$ and if h is proportional to $\bar{g}$, then S is called *totally umbilical* with respect to $\bar{\bar{\nabla}}$ [9].

3. Tangent Bundle of hypersurface with Quarter-Symmetric Metric Connection

Let $\dot{\nabla}$ be a quarter-symmetric metric connection induced on the hypersurface S from $\bar{\nabla}$, which satisfies the connection

$$\nabla_{BX}BY=B(\dot{\nabla}_X Y)+m(X,Y)N, \tag{7}$$

for $X,Y\in\Im_0^1(M)$, where m is a tensor field of type $(0,2)$ in S. Now, by defining $M=H-\eta I$, we obtain the equality

$$m(X,Y)=\bar{g}(MX,Y) \tag{8}$$

for any $X,Y\in\Im_0^1(M)$, where I is the unit tensor field of type $(1,1)$ in S. If m vanishes, then S is called *totally geodesic* with respect to $\dot{\nabla}$ and if m is proportional to $\bar{g}$, then S is called *totally umbilical* with respect to $\dot{\nabla}$.

Theorem 3.1. *The connection induced on a hypersurfaces of a Riemannian manifold with a semi-symmetric metric connection with respect to the unit normal is also a quarter-symmetric metric connection [5].*

Then, we have

$$\dot{\nabla}_X Y = \overline{\overline{\nabla}}_X Y + w(Y)FX - \bar{g}(FX, Y)P \tag{9}$$

for arbitrary $X, Y \in \Im_0^1(M)$. Here P is a vector field in S obtained by $P = BP + \eta N$, where η is a function in S and w is a 1-form in S determined by $\bar{w} = w(BX)$.

For the Riemannian metric g in M, the complete lift $\bar{g}^C$ of $\bar{g}$ is the pseudo-Riemannian metric in TM. Therefore, if we denote the induced metric on TS from $\bar{g}^C$ by $\tilde{g}$, then

$$\tilde{g}(X^C, Y^C) = \bar{g}^C(\tilde{B}X^C, \tilde{B}Y^C) \tag{10}$$

for arbitrary $X, Y \in \Im_0^1(M)$. Thus, the complete lift $\overline{\nabla}^C$ of the Riemannian connection ∇ in (M, g) is the Riemannian connection in the pseudo-Riemannian manifold $(TM, \bar{g}^C)$. Similarly, the complete lift $\overline{\overline{\nabla}}^C$ of the induced connection $\overline{\overline{\nabla}}$ on $(S, \bar{g})$ is also the Riamannian connection in $(TS, \tilde{g})$.

Theorem 3.2. *If T is a torsion tensor of ∇ in (M, g), then T^C is torsion tensor of ∇^C in (TM, g^C) [11].*

Now, the main theorem of this study follows.

Theorem 3.3. *Let $\overline{\nabla}$ be a quarter-symmetric metric connection with respect to Riemannian connection ∇ in (M, g). Then $\overline{\overline{\nabla}}^C$ is also a quarter-symmetric metric connection with respect to ∇^C, where ∇ is the Riemannian connection in (TM, g^C).*

Proof. Firstly, we shall show that $w^V(\tilde{B}X^C) = (w(BX))^V$ and $w^C(\tilde{B}X^C) = (w(BX))^C$. In [10], using (3.10) we get

$$w^V(\tilde{B}X^C) = w^V(BX)^{\bar{C}} = \#(w^V(X^C)) = \#(w(X))^V = (w(BX))\ \bar{V},$$

$$w^C(\tilde{B}X^C) = w^C(BX)^{\bar{C}} = \#(w^C(X^C)) = \#(w(X))^C = (w(BX))\ \bar{C}$$

for arbitrary $X, Y \in \Im_0^1(M)$. Here, we denote the operation of restriction to $\pi_M^{-1}(i(S))$ by $\#$. Also, we denote the vertical and complete lift on $\pi_M^{-1}(i(S))$ by $\bar{V}$ and $\bar{C}$, respectively. Now taking the complete lift of both sides of the equation (2.3) and using equation (3.1) we get

$$(\overline{\nabla}_{BX} BY)^{\bar{C}} = (\nabla_{BX} BY)^{\bar{C}} + (w(BY)F(BX))^{\bar{C}} - (g(FBX, BY)P)^{\bar{C}}$$

$$= (\nabla_{BX} BY)^{\bar{C}} + (w(BY)^{\bar{C}}(F(BX))^{\bar{V}} + (w(BY)^{\bar{V}}(F(BX))^{\bar{C}}$$

$$= (\nabla_{BX} BY)^{\bar{C}} + (w(BY))^{\bar{C}}(F(BX))^{\bar{V}} + (w(BY))^{\bar{V}}(F(BX))^{\bar{C}}$$
$$-(g(FBX, BY))^{\bar{C}} P^{\bar{V}} - (g(FBX, BY))^{\bar{V}} P^{\bar{C}}.$$

$$\overline{\nabla}^C_{\tilde{B}X^C} \tilde{B}Y^C = \nabla^C_{\tilde{B}X^C} \tilde{B}Y^C + w^C(\tilde{B}Y^C)F^C(\tilde{B}X^V) + w^V(\tilde{B}Y^C)F^V(\tilde{B}X^C)$$

$$- g^C(F^C\tilde{B}X^C, \tilde{B}Y^C)P^{\bar{V}} - g^C(F^V\tilde{B}X^V, \tilde{B}Y^C)P^{\bar{C}}.$$

Then, we have

$$\overline{\nabla}^C_{\tilde{B}X^C} \tilde{B}Y^C - \overline{\nabla}^C_{\tilde{B}Y^C} \tilde{B}X^C - [X^C, Y^C] = w^C(\tilde{B}Y^C)F^C(\tilde{B}X^V) + w^V(\tilde{B}Y^C)F^V(\tilde{B}X^C)$$

$$- w^C(\tilde{B}X^C)F^C(\tilde{B}Y^V) - w^V(\tilde{B}X^C)F^V(\tilde{B}Y^C).$$

In view of Theorem 3.2 and (2), we get

$$T^C(\tilde{B}X^C, \tilde{B}Y^C) = w^C(\tilde{B}Y^C)F^C(\tilde{B}X^V) + w^V(\tilde{B}Y^C)F^V(\tilde{B}X^C)$$

$$- w^C(\tilde{B}X^C)F^C(\tilde{B}Y^V) - w^V(\tilde{B}X^C)F^V(\tilde{B}Y^C). \tag{11}$$

By computing, we have

$$g^C(\overline{\nabla}^C_{\tilde{B}X^C} \tilde{B}Y^C, \tilde{B}Z^C) + g^C(\tilde{B}Y^C, \overline{\nabla}^C_{\tilde{B}X^C} \tilde{B}Z^C) = g^C(\nabla^C_{\tilde{B}X^C} \tilde{B}Y^C + w^C(\tilde{B}Y^C)F^C(\tilde{B}X^V) + w^V(\tilde{B}Y^C)F^V(\tilde{B}X^C)$$

$$- g^C(F^C\tilde{B}X^C, \tilde{B}Y^C)P^{\bar{V}} - g^C(F^V\tilde{B}X^V, \tilde{B}Y^C)P^{\bar{C}}, \tilde{B}Z^C))$$

$$+ g^C(\tilde{B}Y^C, \nabla^C_{\tilde{B}X^C} \tilde{B}Z^C + w^C(\tilde{B}Z^C)F^C(\tilde{B}X^V) + w^V(\tilde{B}Z^C)F^V(\tilde{B}X^C)$$

$$-g^C(F^C\tilde{B}X^C,\tilde{B}Z^C)P^{\bar{V}}-g^C(F^V\tilde{B}X^V,\tilde{B}Z^C)P^{\bar{C}}))$$

$$=g^C(\overline{\nabla}^C_{\tilde{B}X^C}\tilde{B}Y^C,\tilde{B}Z^C)+g^C(\tilde{B}Y^C,\overline{\nabla}^C_{\tilde{B}X^C}\tilde{B}Z^C)$$

$$=\left(\tilde{B}X^C\right)g^C(\tilde{B}Y^C,\tilde{B}X^C),$$

we get

$$\left(\overline{\nabla}^C_{\tilde{B}X^C}g^C\right)\left(\tilde{B}Y^C,\tilde{B}Z^C\right)=0. \tag{12}$$

The equation (2.1) and the equation (3.6) implies the required result.

Corollary 3.4. *Let* $\dot{\nabla}$ *be a quarter-symmetirc metric connection with respect to the* $\overline{\overline{\nabla}}$ *Riemannian connection in* $(S,\bar{g})$. *Then,* $\dot{\nabla}^C$ *is also a quarter-symmetric metric connection with respect to* ∇^C *Riemannian connection in* $(TS,\tilde{g})$.

Proof. We have

$$(\overline{\nabla}_{BX}BY)^{\bar{C}}=(\nabla_{BX}BY)^{\bar{C}}+(w(BY)B(FX))^{\bar{C}}-(g(FBX,BY)P)^{\bar{C}}$$

$$(\overline{\nabla}_{BX}BY)^{\bar{C}}=(\nabla_{BX}BY)^{\bar{C}}+(w(BY))(F(BX))^{\bar{C}}-(g(FBX,BY)P)^{\bar{C}}$$

$$=(\nabla_{BX}BY)^{\bar{C}}+(w(BY))^{\bar{C}}(F(BX))^{\bar{V}}+(w(BY))^{\bar{V}}(F(BX))^{\bar{C}}$$

$$-(g(FBX,BY))^{\bar{C}}P^{\bar{V}}-(g(FBX,BY))^{\bar{V}}P^{\bar{C}}.$$

$$\overline{\nabla}^C_{\tilde{B}X^C}\tilde{B}Y^C=\nabla^C_{\tilde{B}X^C}\tilde{B}Y^C+w^C(\tilde{B}Y^C)F^C(\tilde{B}X^V)+w^V(\tilde{B}Y^C)F^V(\tilde{B}X^C)$$

$$-g^C(F^C\tilde{B}X^C,\tilde{B}Y^C)P^{\bar{V}}-g^C(F^V\tilde{B}X^V,\tilde{B}Y^C)P^{\bar{C}},$$

for any $X,Y\in\Im^1_0(M)$. Hence, from the (2.5) and (3.1) we get

$$\left(B(\dot{\nabla}_XY)+m(X,Y)N\right)^{\bar{C}}=(B(\nabla_XY)+h(X,Y)N)\ \bar{C}+w^C(\tilde{B}Y^C)F^C(\tilde{B}X^V)+w^V(\tilde{B}Y^C)F^V(\tilde{B}X^C)$$

$$-g^C(F^C\tilde{B}X^C,\tilde{B}Y^C)(\tilde{B}P^{\bar{V}}+\eta^VN^{\bar{V}})-g^C(F^C\tilde{B}X^V,\tilde{B}Y^C)(\tilde{B}P^{\bar{C}}+\eta^VN^{\bar{C}}+\eta^CN^{\bar{V}}),$$

$$\tilde{B}(\dot{\nabla}_X Y)^C + m^V(X^C, Y^C)N^{\bar{C}} + m^C(X^C, Y^C)N^{\bar{V}}$$

$$= \tilde{B}(\nabla_X Y)^C + h^V(X^C, Y^C)N^{\bar{C}} + h^C(X^C, Y^C)N^{\bar{V}} + w^C(\tilde{B}Y^C)F^C(\tilde{B}X^V) + w^V(\tilde{B}Y^C)F^V(\tilde{B}X^C)$$

$$-g^C(F^C\tilde{B}X^C, \tilde{B}Y^C)\tilde{B}P^V - \eta^V g^C(F^C\tilde{B}X^C, \tilde{B}Y^C)N\ \bar{V}$$

$$-g^C(F^C\tilde{B}X^C, \tilde{B}Y^C)\tilde{B}P^C - \eta^V g^C(F^C\tilde{B}X^C, \tilde{B}Y^C)N\ \bar{C}$$

$$-\eta^C g^C(F^C\tilde{B}X^V, \tilde{B}Y^C)N\ \bar{V}.$$

Moreover, we get

$$\tilde{B}(\dot{\nabla}_X Y)^C = \tilde{B}(\nabla_X Y)^C + w^C(\tilde{B}Y^C)F^C(\tilde{B}X^V) + w^V(\tilde{B}Y^C)F^V(\tilde{B}X^C) \quad (13)$$

$$-g^C(F^C\tilde{B}X^C, \tilde{B}Y^C)\tilde{B}P^V - g^C(F^C\tilde{B}X^V, \tilde{B}Y^C)\tilde{B}P^C,$$

and

$$m^V(X^C, Y^C)N^{\bar{C}} + m^C(X^C, Y^C)N^{\bar{V}} = (h^V(X^C, Y^C) \quad (14)$$

$$-\eta^V g^C(F^C\tilde{B}X^C, \tilde{B}Y^C)N\ \bar{C}$$

$$+(h^C(X^C, Y^C) - \eta^V g^C(F^C\tilde{B}X^C, \tilde{B}Y^C) - \eta^C g^C(F^C\tilde{B}X^V, \tilde{B}Y^C))N\ \bar{V}.$$

From (3) and (4), it follows that

$$(\dot{\nabla}_X Y)^C = (\nabla_X Y)^C + w^C(Y^C)F^C X^V + w^V(Y^C)F^V X^C - \tilde{g}(F^C X^C, Y^C)P^V - \tilde{g}(F^V X^V, Y^C)P^C.$$

Finally, we obtain

$$\dot{\nabla}^C_{X^C} Y^C = \nabla^C_{X^C} Y^C + w^C(Y^C)F^C X^V + w^V(Y^C)F^V X^C - \tilde{g}(F^C X^C, Y^C)P^V - \tilde{g}(F^V X^V, Y^C)P^C.$$

Thus, we have

$$\dot{\nabla}^C_{X^C} Y^C - \dot{\nabla}^C_{Y^C} X^C - [X^C, Y^C] = w^C(Y^C)F^C X^V + w^V(Y^C)F^V X^C - w^C(X^C)F^C Y^V - w^V(X^C)F^V Y^C,$$

this implies

$$\dot{T}(X^C, Y^C) = w^C(Y^C)F^C X^V + w^V(Y^C)F^V X^C - w^C(X^C)F^C Y^V \quad (15)$$

$$-w^V(X^C)F^V Y^C.$$

Similarly, we have

$$\tilde{g}(\dot{\nabla}^C_{X^C} Y^C, Z^C) + \tilde{g}(Y^C, \dot{\nabla}^C_{X^C} Y^C) = X^C(\tilde{g}(Y^C, Z^C)),$$

$$\left(\dot{\nabla}^C_{X^C}\tilde{g}\right)(Y^C, Z^C) = 0. \tag{16}$$

The equation (3.9) and the equation (3.10) completes the proof of the theorem.

The Quarter-symmetric metric connection $\dot{\nabla}^C$ on $(TS, \tilde{g})$ can be given by

$$\dot{\nabla}^C_{X^C} Y^C = \nabla^C_{X^C} Y^C + w^C(Y^C)F^C X^V + w^V(Y^C)F^V X^C - \tilde{g}(F^C X^C, Y^C)P^V - \tilde{g}(F^V X^V, Y^C)P^C.$$

Taking the complete lift of both side of the equation (3.1), we get

$$\overline{\nabla}^C_{\tilde{B}X^C} \tilde{B}Y^C = \tilde{B}(\dot{\nabla}^C_{X^C} Y^C) + m^V(X^C, Y^C)N^{\bar{C}} + m^C(X^C, Y^C)N^{\bar{V}}.$$

From (3.1) , (3.2) and (3.8), it follows that

$$and m^V(X^C, Y^C) = h^V(X^C, Y^C) - \eta^V g^C(F^C \tilde{B}X^V, \tilde{B}Y^C),$$

$$m^C(X^C, Y^C) = h^C(X^C, Y^C) - \eta^V g^C(F^C \tilde{B}X^C, \tilde{B}Y^C) - \eta^C g^C(F^V \tilde{B}X^V, \tilde{B}Y^C).$$

According to [10], TS is totally umbilical if and only if there exist differentiable functions λ and μ, such that

$$m^V(\tilde{X}, \tilde{Y}) = \lambda\tilde{g}(\tilde{X}, \tilde{Y}),$$

and

$$m^C(\tilde{X}, \tilde{Y}) = \mu\tilde{g}(\tilde{X}, \tilde{Y}),$$

for any $\tilde{X}, \tilde{Y} \in \Im^1_0(TS)$. If both λ and μ vanish, then TS is totally geodesic.

It is trivial to prove the following theorems by using (3.1) and (3.2).

Theorem 3.5. TS *is totally umbilical with respect to the quarter-symmetric metric connection* $\dot{\nabla}^C$ *if and only if* S *is totally umbilical or totally geodesic with respect to the Riemannian connection* ∇^C.

Theorem 3.6. TS *is totally umbilical with respect to the quarter-symmetric metric connection* $\dot{\nabla}^C$ *if and only if* S *is totally umbilical with respect to the quarter-symmetric metric connection* $\dot{\nabla}$.

Theorem 3.7. TS *is totally geodesic with respect to the quarter-symmetric metric connection* $\dot{\nabla}^C$ *if and only if it is totally geodesic with respect to the Riemannian connection* ∇^C *and the vector field* P *is tangent to* S.

Theorem 3.8. *TS is totally geodesic with respect to the quarter-symmetric metric connection $\dot{\nabla}^C$ if and only if S is totally geodesic with respect to the quarter-symmetric metric connection $\dot{\nabla}$.*

4. The Structure Equations of Tangent Bundle with Quarter-Symmetric Metric Connection

Theorem 4.1. *In view of [1], the structure equations of S with respect to the quarter-symmetric metric connection are given by:*

$$\overline{\nabla}_{BX} N = -BMX + w(N)F(BX) - g(FBX, N)P,$$

$$\overline{g}(\dot{R}(X,Y)Z,W) = g((\overline{R}(BX,BY)BZ,BW) + \overline{g}((MX)m(Y,Z) - (MY)m(X,Z),W)$$

$$+ g((\dot{\nabla}_X m)(Y,Z) - (\dot{\nabla}_Y m)(X,Z),W)) + g(m(w(Y)FX - w(X)FY,Z)N,W)),$$

$$g(\overline{R}(BX,BY)N,BZ) = \overline{g}(\dot{\nabla}_Y MX - \dot{\nabla}_X MY + M[X,Y],Z),$$

for any $X,Y,Z \in \Im_0^1(S)$.

Theorem 4.2. *If R is the curvature tensor field of the Riemannian connection ∇ in (M,g). Then, the complete lift R^C of R is the curvature tensor field of the Riemnnian connection ∇^C in (TM, g^C). Similarly, the complete lift R^C of R is the curvature tensor field of the Riemannian connection $\overline{\overline{\nabla}}^C$ in $(TS,\tilde{g})$, where R is the curvature tensor field of the induced connection $\overline{\overline{\nabla}}$ on $(S,\overline{g})$ [9].*

Let $\overline{R}$ be a curvature tensor field of the quarter-symmetric metric connection $\overline{\nabla}$ in (M,g). Then, the curvature tensor field of the quarter symmetric metric connection $\overline{\nabla}^C$ is $\overline{R}^C$ in (TM, g^C). Similarly, the complete lift $\dot{R}^C$ of $\dot{R}$ is the curvature tensor field of the quarer-symmetric metric connection $\dot{\nabla}^C$ in $(TS,\tilde{g})$ where $\dot{R}$ is the curvature tensor field of the induced connection $\dot{\nabla}$ on $(S,\overline{g})$.

Theorem 4.3. *In [11] , The Weingarten eqution of TS are given by*

$$\overline{\nabla}^C_{\tilde{B}X^C} N^{\overline{V}} = -\tilde{B}M^V X^C,$$

$$\overline{\nabla}^C_{\tilde{B}X^C} N^{\overline{C}} = -\tilde{B}M^C X^C,$$

for any $X, Y, \in \Im_0^1(S)$.

Theorem 4.4. *The Weingarten equation of* TS *with respect to the quarter-symmetric metric connection are*

$$\overline{\nabla}_{\tilde{B}}^{V} X^C N^{\bar{V}} = -\tilde{B} M^V X^C + w^V (N^V F^V \tilde{B} X^C - g(F^V \tilde{B} X^C, N^V) P^V \tag{17}$$

$$\overline{\nabla}_{\tilde{B}}^{C} X^C N^{\bar{V}} = -\tilde{B} M^C X^C + w^C (N^V F^C \tilde{B} X^C - g(F^V \tilde{B} X^C, N^C) P^C \tag{18}$$

Proof. Using equation(2.3), Theorem 9 and by the virtue of section 2 in [10], we get equations (4.1) and (4.2).

Theorem 4.5. *The Gauss equation with respect to the quarter-symmetric metric connection of* TS *is obtained as:*

$$\tilde{g}(\dot{R}^C(X^C, Y^C) Z^C, W^C) = g^C(\overline{R}^C(\tilde{B} X^C, \tilde{B} Y^C) \tilde{B} Z^C, \tilde{B} W^C) \tag{19}$$

$$+\tilde{g}((M^C X^C) m^V (Y^C, Z^C) + (M^V X^C) m^C (Y^C, Z^C), W^C)$$

$$-\tilde{g}((M^C Y^C) m^V (X^C, Z^C) + (M^V Y^C) m^C (X^C, Z^C), W^C)$$

$$+\tilde{g}\{((\dot{\nabla}_{Y^C} m^V)(X^C, Z^{C)}) - (\dot{\nabla}_{Y^C} m^V)(Y^C, Z^{C)}), W^{C)})\} N^{\bar{V}}$$

$$\tilde{g}\{((+\tilde{g}\{((\dot{\nabla}_{Y^C} m^C)(X^C, Z^{C)}) - (\dot{\nabla}_{Y^C} m^C)(Y^C, Z^{C)}), W^{C)})\} N^{\bar{C}}$$

$$+\tilde{g}\{((m^V w^V (Y^C)(FX)^V - (m^V w^V (X^C)(FY)^V, Z^C)\} N^V, W^C)$$

$$-\tilde{g}\{((m^C w^C (Y^C)(FX)^C - (m^C w^C (X^C)(FY)^C, Z^C)\} N, W^C)$$

for any $X, Y, Z \in \Im_0^1(S)$.

Proof. Using the equation (4) , Theorem 9 and by the virtue of the section 2 nd the equation (6.4) in [10], we get,

$$\tilde{g}(\dot{R}^C(X^C, Y^C) Z^C, W^C) = \tilde{g}((\dot{R}(X, Y) Z)^C, W^C)$$

$$= (g(\overline{R}(BX, BY) BZ, BW) + g((MX) m(Y, Z) - (MY)(X, Z), W)$$

$$+ g\{((\dot{\nabla}_Y m)(X, Z) - (\dot{\nabla}_X m)(Y, Z)\} N, W) + g((m(w(Y) FX - w(X) FY, Z) N, W))^C$$

after solving, we get the desired equation (19).

Theorem 4.6. *In [11], The Codazz-Ricci equations of TS are obtained as:*

$$\bar{R}^C(\tilde{B}X^C, \tilde{B}Y^C)N^{\bar{V}} = \tilde{B}(\dot{\nabla}_{Y^C} M^V X^C - \dot{\nabla}_{X^C} M^V Y^C + M^V[X^C, Y^C])$$

$$\bar{R}^C(\tilde{B}X^C, \tilde{B}Y^C)N^{\bar{C}} = \tilde{B}(\dot{\nabla}_{Y^C} M^C X^C - \dot{\nabla}_{X^C} M^C Y^C + M^V[X^C, Y^C])$$

$$\bar{R}^C(N^{\bar{V}}, N^{\bar{C}})\tilde{B}X^C = 0,$$

for any $X, Y, Z \in \Im_0^1(S)$.

Theorem 4.7. *The Codazzi-Ricci equations with respect to the quarter-symmetric metric connection of TS are obtained as:*

$$\bar{R}^C(\tilde{B}X^C, \tilde{B}Y^C)N^{\bar{V}} = \tilde{B}(\dot{\nabla}^C_{Y^C} M^V X^C - \dot{\nabla}^C_{X^C} M^V Y^C + M^V[X^C, Y^C]) \tag{20}$$

$$+w^V(N^V)[\tilde{B}(\dot{\nabla}^C_{X^C} F^V Y^C - \dot{\nabla}^C_{Y^C} F^V Y^C + F^V[X^C, Y^C])]$$

$$+[(\dot{\nabla}^C_{\tilde{B}Y^C} w^V - \dot{\nabla}^C_{\tilde{B}X^C} w^V - w^C(Y^C)F^V(\tilde{B}X^C - w^V(X^C)F^V(\tilde{B}Y^C)]N^V)$$

$$+\tilde{B}[w^V(M^V Y^C) - w^V(M^V X^C) + w^V(N^V)w^V(F^V X^C) - w^V(N^V)w^V(F^V Y^C)$$

$$+\tilde{g}(F^V X^C, N^V)w^V(F^V Y^C) - \tilde{g}(F^V Y^C, N^V)w^V(F^V X^C)]P^V$$

$$\bar{R}^C(\tilde{B}X^C, \tilde{B}Y^C)N^{\bar{C}} = \tilde{B}(\dot{\nabla}^C_{Y^C} M^C X^C - \dot{\nabla}^C_{X^C} M^C Y^C + M^C[X^C, Y^C]) \tag{21}$$

$$+w^C(N^C)[\tilde{B}(\dot{\nabla}^C_{X^C} F^C Y^C - \dot{\nabla}^C_{Y^C} F^C Y^C + F^C[X^C, Y^C])]$$

$$+[(\dot{\nabla}^C_{\tilde{B}Y^C} w^C - \dot{\nabla}^C_{\tilde{B}X^C} w^C - w^C(Y^C)F^V(\tilde{B}X^C - w^C(X^C)F^C(\tilde{B}Y^C)]N^C)$$

$$+\tilde{B}[w^C(M^C Y^C) - w^C(M^C X^C) + w^C(N^C)w^C(F^C X^C) - w^V(N^C)w^C(F^C Y^C)$$

$$+\tilde{g}(F^C X^C, N^C) w^C (F^C Y^C) - \tilde{g}(F^C Y^C, N^C) w^C (F^C X^C)] P^C$$

$$\overline{R}^C (N^{\overline{V}}, N^{\overline{C}}) \tilde{B} X^C = w^C (Y^C) F^C (N^C) - w^C (X^C) F = 0, \tag{22}$$

for any $X, Y, Z \in \Im_0^1(S)$.

Proof. From equation (4), (17), (18) and theorem 9,we get the equations (20), (21) and (22).

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